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2S HYPERFINE SPLITTING OF MUONIC HYDROGEN

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Corrections of orders α^5, α^6 are calculated in the hyperfine splitting of the $2S$ state in the muonic hydrogen. The nuclear structure effects are taken into account in the one- and two-loop Feynman amplitudes by means of the proton electromagnetic form factors. Total numerical value of the $2S$ state hyperfine splitting 22.8148 meV in the μp can be considered as reliable estimation for the corresponding experiment with the accuracy 10^{-5}. The value of the Sternheim's hyperfine splitting interval $[8\Delta E^{HFS}(2S)-\Delta E^{HFS}(1S)]$ is obtained with the accuracy 10^{-6}.

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I. INTRODUCTION

The investigation of the energy levels of hydrogenic atoms with high accuracy is important for the check of the Standard Model and extraction the values of fundamental physical constants (the fine structure constant, the electron and muon masses, the proton charge radius etc.) [1]. The Zemach radius takes substantial role among many fundamental parameters determining the structure of the energy levels of simple atomic systems [2]. It is just this parameter which is responsible for the hyperfine splitting (HFS) interval in electronic and muonic hydrogen. The ground state hydrogen hyperfine splitting measurement was made many years ago with very high accuracy [3]:

$$\Delta\nu^{HFS}_{exp}(ep) = 1\ 420\ 405.751\ 766\ 7(9)\ kHz. \tag{1}$$

The theoretical expression for the hyperfine splitting in the hydrogen

$$\Delta E^{HFS}_{theor} = E^F\left(1+\delta^{QED}+\delta^{str}+\delta^{pol}+\delta^{HVP}\right),\quad E^F=\frac{8}{3}\alpha^4\frac{\mu_P m_1^2 m_2^2}{(m_1+m_2)^3}, \tag{2}$$

includes several corrections to the Fermi energy E^F: δ^{QED} represents the QED contribution, the corrections δ^{str} and δ^{pol} are the proton structure and polarizability contributions, δ^{HVP} is the contribution of hadronic vacuum polarization (HVP). μ_p is the proton magnetic moment in nuclear magnetons, m_1 is the lepton mass, m_2 is the proton mass. The expression (2) is valid both for the muonic and electronic hydrogen but the exact value of the corrections is

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TABLE I: The values of the Zemach radii.

The Zemach radius	Hyperfine splitting in electronic hydrogen [21]	Hyperfine splitting in electronic hydrogen [22]	Elastic e-p scattering [23]	Elastic e-p scattering [24]
R_p	1.047(19) fm	1.040(16) fm	1.086(12) fm	1.013(16) fm

essentially different for these atoms. The proton structure and polarizability effects lead to main theoretical uncertainty in the expression (2). The investigation of the nuclear structure and polarizability corrections in the hyperfine splitting of the energy levels was done in many papers [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] (see complete discussion in the review article [17]). It is valid to say that during several tens of years the theory fall behind the experiment in solving this task. Existing difference between the theory and experiment without accounting the proton polarizability contribution can be expressed as follows [17]:

$$\frac{\Delta E^{HFS}_{theor}(e\ p)-\Delta E^{exp}_{HFS}(e\ p)}{E^F(e\ p)}=-4.5(1.1)\times 10^{-6}. \tag{3}$$

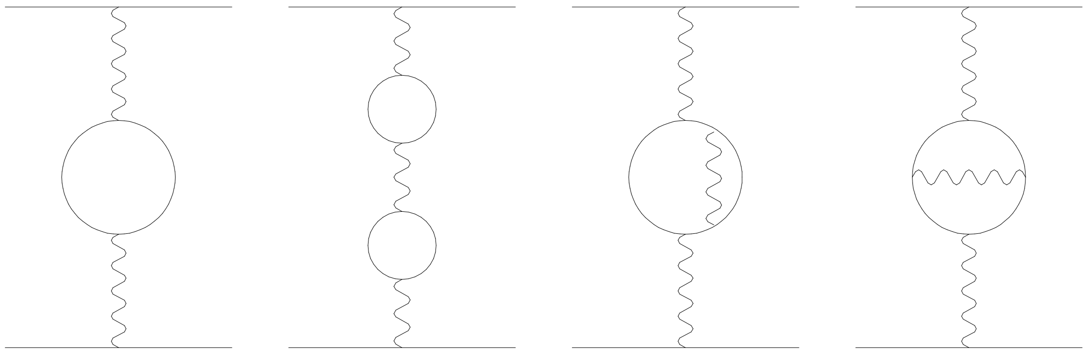
There exists at least two possibilities to improve theoretical expression (2). First possibility is connected with new more exact measurement of the electric G_E and magnetic G_M proton form factors. The integral over the product of these form factors gives main part of the proton structure correction δ^{str} (the Zemach correction [2]):

$$\Delta E_Z=E^F\frac{2\mu\alpha}{\pi^2}\int\frac{d\mathbf{p}}{\mathbf{p}^4}\left[\frac{G_E(-\mathbf{p}^2)G_M(-\mathbf{p}^2)}{\mu_P}-1\right]=E^F(-2\mu\alpha)R_p,\ W=\alpha\mu, \tag{4}$$

where μ is the reduced mass of two particles, R_p is the Zemach radius. In the coordinate representation the Zemach correction (4) is determined by the contraction of the charge ρ_E and magnetic moment ρ_M distributions. The Zemach radius represents the integral characteristic of the proton structure effects in the hyperfine splitting of the energy levels. It may be considered as new fundamental proton parameter in the hydrogen atom.

Another possibility consists in the using the muonic hydrogen. The measurement of the hyperfine splitting of the $1S$ and $2S$ energy levels with the accuracy 10^{-5} in the muonic hydrogen [18, 20, 21] allows to obtain the value of the Zemach radius with the accuracy 10^{-3}. Then it can be used for obtaining new theoretical value for the hyperfine splitting in the electronic hydrogen and new theoretical restriction on the value of the proton polarizability contribution. An attempts to obtain the value of the Zemach radius were made recently using just as the experimental data on elastic electron-proton scattering so the hyperfine splitting in the hydrogen atom (see the Table 1 and the references [21, 22, 23, 24]). The difference between corresponding results reaches 7%.

The investigation of different contributions to the energy levels of the muonic atoms was done many years ago in Ref. [25, 26, 27]. So, at present there is need for new more complete analysis of all possible corrections in the HFS of the μp with the declared accuracy 10^{-5}. The corrections of order α^5 to the hyperfine splitting of the $2S$ state and the Lamb shift $2P-2S$ in the μp were calculated in Refs.[27, 28]. In this study we continue the investigation of the different contributions of orders α^5 and α^6 to the muonic hydrogen HFS of the S states which was begun in Ref.[29]. The aim of the work consists in obtaining the numerical value of the $2S$ HFS with the accuracy 10^{-5} and the Sternheim's hyperfine splitting interval



a b c d

FIG. 1: Effects of the one- and two-loop vacuum polarization in the one-photon interaction.

$[8\Delta E^{HFS}(2S)-\Delta E^{HFS}(1S)]$ [30] in the muonic hydrogen with the accuracy 10^{-6} which can serve as reliable guide for corresponding experiment and the measurement $2P-2S$ Lamb shift [31]. Basic problems of the HFS measurement in the muonic hydrogen were discussed in Refs.[20, 21].

II. EFFECTS OF VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Our calculation of different energy levels of the hydrogen-like atoms are carried out on the basis of the quasipotential approach where the two-particle bound state is described by the Schroedinger-type equation [32]:

$$\left[G^f\right]^{-1}\psi_M \equiv \left(\frac{b^2}{2\mu_R}-\frac{\mathbf{p}^2}{2\mu_R}\right)\psi_M(\mathbf{p})=\int\frac{d\mathbf{q}}{(2\pi)^3}V(\mathbf{p},\mathbf{q},M)\psi_M(\mathbf{q}), \tag{5}$$

where

$$b^2=E_1^2-m_1^2=E_2^2-m_2^2,$$

$\mu_R=E_1E_2/M$ is the relativistic reduced mass, $M=E_1+E_2$ is the bound state mass. The quasipotential of the equation (5) is constructed in the quantum electrodynamics by the perturbative series using projected on positive states the two-particle off mass shell scattering amplitude T at zero relative energies of the particles:

$$V=V^{(1)}+V^{(2)}+V^{(3)}+...,\qquad T=T^{(1)}+T^{(2)}+T^{(3)}+..., \tag{6}$$

$$V^{(1)}=T^{(1)},\ V^{(2)}=T^{(2)}-T^{(1)}\times G^f\times T^{(1)},...\ . \tag{7}$$

We take the ordinary Coulomb potential as initial approximation for the quasipotential $V(\mathbf{p},\mathbf{q},M)$: $V(\mathbf{p},\mathbf{q},M)=V^C(\mathbf{p}-\mathbf{q})+\Delta V(\mathbf{p},\mathbf{q},M)$.

The increase of the lepton mass when we change the electronic hydrogen to the muonic hydrogen leads to the decrease of the Bohr radius in the μp. As a result the electron Compton

wave length and the Bohr radius are of the same order:

$$\frac{\hbar^2}{\mu e^2}:\frac{\hbar}{m_e c}=0.737384 \tag{8}$$

(m_e is the electron mass, μ is the reduced mass in the atom μp). An important consequence of last relation is the increase the role of the electron vacuum polarization effects in the energy spectrum of the μp [33]. The effects of the vacuum polarization in the one-photon interaction are shown in Fig.1.

To obtain the contribution of the diagram (a) Fig.1 (the electron vacuum polarization) to the interaction operator there is need to make the following substitution in the photon propagator [33]:

$$\frac{1}{k^2}\to\frac{\alpha}{\pi}\int_0^1 dv\frac{v^2\left(1-\frac{v^2}{3}\right)}{k^2(1-v^2)-4m_e^2}. \tag{9}$$

At $(-k^2)=\mathbf{k}^2\sim\mu_e^2(Z\alpha)^2\sim m_e^2(Z\alpha)^2$ (electronic hydrogen, μ_e is the reduced mass in hydrogen atom) we obtain $-\alpha/15\pi m_e^2$ omitting the first term in the denominator of right part of Eq.(9). But when $\mathbf{k}^2\sim\mu^2(Z\alpha)^2\sim m_1^2(Z\alpha)^2$ (muonic hydrogen, m_1 is the muon mass) than $\mu\alpha$ and m_e are of the same order and it is impossible to use expansion over α in the denominator of Eq.(9). To construct the hyperfine part of the quasipotential in this case (the muonic hydrogen) in the one-photon interaction we must use exact expression (9). We take into account that the appearance of the electron mass m_e in the denominator of the amplitude leads effectively to the decrease the order of the correction. It is well known that the hyperfine splitting quasipotential has the form [34]:

$$V_{1\gamma}^{HFS}(\mathbf{k})=\frac{4\pi Z\alpha}{m_1m_2}\frac{1+\kappa}{4}\frac{1}{\mathbf{k}^2}[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\mathbf{k}^2-(\boldsymbol{\sigma}_1\mathbf{k})(\boldsymbol{\sigma}_2\mathbf{k})]. \tag{10}$$

For the S-states

$$V_{1\gamma}^{HFS}(\mathbf{k})=\frac{8\pi Z\alpha}{3m_1m_2}\frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{4}(1+\kappa), \tag{11}$$

κ=1.792847337(29) is the proton anomalous magnetic moment. Averaging the potential (11) over the Coulomb wave functions we obtain main contribution of order $(Z\alpha)^4$ to the HFS of the $2S$-state in the system μp (the Fermi energy):

$$E^F(2S)=\frac{1}{3}(Z\alpha)^4\frac{\mu^3}{m_1m_2}(1+\kappa)=22.8054\ meV. \tag{12}$$

The modification of the Coulomb potential due to the vacuum polarization (VP) is determined by means of Eq.(9) in the momentum representation as follows [33]:

$$V_{VP}^C(\mathbf{k})=-4\pi Z\alpha\frac{\alpha}{\pi}\int_1^\infty\frac{\sqrt{\xi^2-1}}{3\xi^4}\frac{(2\xi^2+1)}{\mathbf{k}^2+4m_e^2\xi^2}d\xi \tag{13}$$

In the coordinate representation we obtain:

$$V_{VP}^C(r)=\frac{\alpha}{3\pi}\int_1^\infty d\xi\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}\left(-\frac{Z\alpha}{r}e^{-2m_e\xi r}\right). \tag{14}$$

The contribution of the electron vacuum polarization to the hyperfine splitting part of the 1γ quasipotential for the S-states can be derived in a similar way in the momentum and coordinate representations:

$$V^{HFS}_{1\gamma,\ VP}(\mathbf{k})=\frac{8\pi Z\alpha(1+\kappa)}{3m_1m_2}\frac{(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)}{4}\mathbf{k}^2\frac{\alpha}{\pi}\int_1^\infty\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{3\xi^4(\mathbf{k}^2+4m_e^2\xi^2)}d\xi, \tag{15}$$

$$V^{HFS}_{1\gamma,\ VP}(r)=\frac{8Z\alpha(1+\kappa)}{3m_1m_2}\frac{(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)}{4}\frac{\alpha}{\pi}\int_1^\infty\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{3\xi^4}d\xi\left[\pi\delta(\mathbf{r})-\frac{m_e^2\xi^2}{r}e^{-2m_e\xi r}\right]. \tag{16}$$

Using Eq.(16) we can obtain the electron vacuum polarization correction of order α^5 to the HFS in the μp. Taking the wave function of the $2S$-state

$$\psi_{200}(r)=\frac{W^{3/2}}{2\sqrt{2\pi}}e^{-Wr/2}\left(1-\frac{Wr}{2}\right),\quad W=\mu Z\alpha, \tag{17}$$

we represent this correction in the form:

$$\Delta E_{1\gamma,VP}HFS=\frac{\mu^3(Z\alpha)^4(1+\kappa)}{3m_1m_2}\frac{\alpha}{\pi}\frac{m_e^3}{3W^3}\int_{m_e/W}^\infty\frac{\sqrt{\frac{W^2}{m_e^2}\xi^2-1}}{\xi^4}\left(2\frac{W^2}{m_e^2}\xi^2+1\right)d\xi\,\times \tag{18}$$
$$\times\left[1-\int_0^\infty e^{-r(2\xi+1)/2\xi}\left(1-\frac{r}{4}\right)^2 rdr\right]=0.0481\ meV.$$

The contribution of the muon vacuum polarization (MVP) can be found by means (16) after the substitution $m_e\to m_1$. This correction is of order α^6 due to the reason mentioned above. Numerical value is equal

$$\Delta E^{HFS}_{1\gamma,\ MVP}=E^F(2S)\frac{3}{8}\frac{\mu}{m_1}Z\alpha^2=0.0004\ meV. \tag{19}$$

In this expression and below $E^F(2S)$ is the Fermi energy of the $2S$ state. The diagrams of the two-loop electron vacuum polarization shown in Fig.1 (b,c,d) give the contributions of the same order α^6. The interaction operator corresponding to the loop after loop amplitude can be obtained using the relation (9). In the coordinate representation

$$V^{HFS}_{1\gamma,\ VP-VP}(r)=\frac{8\pi Z\alpha(1+\kappa)}{3m_1m_2}\frac{(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)}{4}\left(\frac{\alpha}{\pi}\right)^2\int_1^\infty\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{3\xi^4}d\xi\,\times \tag{20}$$
$$\times\int_1^\infty\frac{\sqrt{\eta^2-1}(2\eta^2+1)}{3\eta^4}d\eta\left[\delta(\mathbf{r})-\frac{m_e^2}{\pi r(\eta^2-\xi^2)}\left(\eta^4e^{-2m_e\eta r}-\xi^4e^{-2m_e\xi r}\right)\right],$$

and the contribution to the energy spectrum

$$\Delta E^{HFS}_{1\gamma,\ VP-VP}=0.0002\ meV. \tag{21}$$

To calculate the contributions of the diagrams b, c in Fig.1 which are determined by the polarization operator of the second order it is necessary to make the substitution in the photon propagator [35]:

$$\frac{1}{k^2}\to\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)}{4m_e^2+k^2(1-v^2)}dv=\left(\frac{\alpha}{\pi}\right)^2\frac{2}{3}\int_0^1 dv\frac{v}{4m_e^2+k^2(1-v^2)}\,\times \tag{22}$$

$$\times\left\{(3-v^2)(1+v^2)\left[Li_2\left(-\frac{1-v}{1+v}\right)+2Li_2\left(\frac{1-v}{1+v}\right)+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}-\ln\frac{1+v}{1-v}\ln v\right]+\right.$$
$$\left.\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]+\frac{3}{8}v(5-3v^2)\right\}.$$

To find numerical value of this correction we write the quasipotential in the coordinate space:

$$\Delta V^{HFS}_{1\gamma,\ 2-loop\ VP})=\frac{8\pi Z\alpha(1+\kappa)}{3m_1m_2}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{(1-v^2)}\left[\delta(\mathbf{r})-\frac{m_e^2}{\pi r(1-v^2)}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\right].\tag{23}$$

The potential (23) gives the contribution to the $2S$ HFS in the muonic hydrogen

$$\Delta E^{HFS}_{1\gamma,\ 2-loop\ VP}=0.0001\ meV.\tag{24}$$

We calculate all contribution numerically and the results are presented with the accuracy 0.0001 meV.

III. SECOND ORDER OF THE PERTURBATION THEORY

The corrections of the second order of the perturbative series in the energy spectrum are defined by the reduced Coulomb Green function (RCGF) [36]:

$$\tilde{G}_2(\mathbf{r},\mathbf{r}')=\sum_{l,m}\tilde{g}_{nl}(r,r')Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}').\tag{25}$$

The radial wave function $\tilde{g}_{nl}(r,r')$ was obtained in Ref.[36] as an expansion over the Laguerre polynomials. For the $2S$ - state

$$\tilde{g}_{20}(r,r')=-2\mu^2Z\alpha\left[e^{-\frac{x+x'}{2}}\sum_{m=1,m\neq 2}^{\infty}\frac{L^1_{m-1}(x)L^1_{m-1}(x')}{m(m-2)}+\left(\frac{5}{2}+x\frac{\partial}{\partial x}+x'\frac{\partial}{\partial x'}\right)e^{-\frac{x+x'}{2}}L^1_1(x)L^1_1(x')\right],\tag{26}$$

where $x=\mu Z\alpha r$, L^m_n are the Laguerre polynomials:

$$L^m_n(x)=\frac{e^xx^{-m}}{n!}\left(\frac{d}{dx}\right)^n\left(e^{-x}x^{n+m}\right).\tag{27}$$

Some terms of the quasipotential contain the $\delta(\mathbf{r})$ so we have to know the quantity $\tilde{G}_2(\mathbf{r},0)$. The expression for the RCGF was found in this case in Ref.[37] on the basis of the Hoestler representation for the Coulomb Green function after the subtraction the pole term:

$$\tilde{G}_{2S}(\mathbf{r},0)=-\frac{Z\alpha\mu^2}{4\pi}\frac{e^{-x/2}}{2x}\left[4x(x-2)(\ln x+C)+x^3-13x^2+6x+4\right],\tag{28}$$

where $C=0.5772...$ is the Euler constant. The main contribution of order α^5 in the second order of the perturbation theory can be written in general form:

$$\Delta E^{HFS}_{1\ SOPT}=\sum_{n=1,n\neq 2}^{\infty}\frac{<\psi^c_2|V^C_{VP}|\psi^c_n><\psi^c_n|\Delta V^{HFS}_{1\gamma}|\psi^c_2>}{E^c_2-E^c_n},\tag{29}$$

where $\Delta V_{1\gamma}^{HFS} \sim \delta(\mathbf{r})$. Using the relations (14), (28) we can present Eq.(29) as follows:

$$\Delta E_{1~SOPT}^{HFS} = E^F(2S)(1+a_\mu)\frac{\alpha}{3\pi}\int_1^\infty d\xi \frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}\times \tag{30}$$

$$\times \int_0^\infty dx e^{-x(1+\frac{2m_e\xi}{W})}\left[4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4\right] = 0.0746~meV.$$

The contribution of order α^6 in the second order of the perturbative series which is determined by the vacuum polarization can be derived from Eq.(29) changing $\Delta V_{1\gamma}^{HFS} \to \Delta V_{1\gamma~VP}^{HFS}$. Using exact expressions for the wave function $\psi_2^c(\mathbf{r})$ (17) and the RCGF (28) we write this correction

$$\Delta E_{2~SOPT}^{HFS} = E^F(2S)\frac{\alpha^2}{9\pi^2}\int_1^\infty d\xi \frac{\sqrt{\xi^2-1}(1+2\xi^2)}{\xi^4}\times \tag{31}$$

$$\times \int_1^\infty d\eta \frac{\sqrt{\eta^2-1}(1+2\eta^2)}{\eta^4}\left[F_1(\xi)+F_2(\xi,\eta)+F_3(\xi,\eta)\right],$$

$$F_1(\xi) = \frac{1}{a_1^5}\left[-12+23a_1-8a_1^2-4a_1^3+4a_1^4+4a_1(3-4a_1+2a_1^2)\ln a_1\right], a_1 = \left(1+\frac{2m_e}{W}\xi\right), \tag{32}$$

$$F_2(\xi,\eta) = -\frac{2m_e^2\xi^2}{W^2}\frac{1}{a_2^5b_2^5}\Bigl[-12b_2(3-4b_2+2b_2^2)+6a_2^2(-4+11b_2-8b_2^2+2b_2^3)-4a_2^2(-12+39b_2-$$

$$-34b_2^2+12b_2^3)+3a_2(-12+51b_2-52b_2^2+22b_2^3)\Bigr],~a_2 = a_1, b_2 = \left(1+\frac{2m_e}{W}\eta\right),$$

$$F_3(\xi,\eta) = -\frac{16m_e^2\xi^2}{W^2}\frac{a_3}{\left(1+\frac{2m_e\xi}{W}\right)^3\left(1+\frac{2m_e\eta}{W}\right)^3}\Bigl\{\frac{4m_e^2}{W^2}\xi\eta\left[\frac{a_3}{1-a_3}-2\ln(1-a_3)\right]+$$

$$+\frac{1}{2}\left(\frac{\xi}{\eta}+\frac{\eta}{\xi}\right)\left[\frac{4a_3}{1-a_3}+\frac{a_3^2}{(1-a_3)^2}-2\ln(1-a_3)\right]+\frac{W^2}{16m_e^2\xi\eta}\Bigl[\frac{19a_3}{1-a_3}+\frac{6a_3^2}{(1-a_3)^2}+$$

$$+\frac{a_3^2(1+a_3)}{(1-a_3)^3}-2\ln(1-a_3)\Bigr]\Bigr\},~a_3 = \frac{1}{\left(1+\frac{W}{2m_e\xi}\right)\left(1+\frac{W}{2m_e\eta}\right)}.$$

Numerical value of this contribution is equal

$$\Delta E_{2~SOPT}^{HFS} = 0.0003~meV. \tag{33}$$

The second order of the perturbative series gives also other relativistic corrections of order $(Z\alpha)^6$ including recoil effects which were studied in Ref.[38, 39, 40]. Corresponding numerical data are in the Table 1.

IV. PROTON STRUCTURE AND VACUUM POLARIZATION EFFECTS

The proton structure corrections in the system μp are relatively large in the comparison with the electronic hydrogen. In the HFS of the muonic hydrogen these corrections are defined in the leading order by the one-loop diagrams in Fig.2.

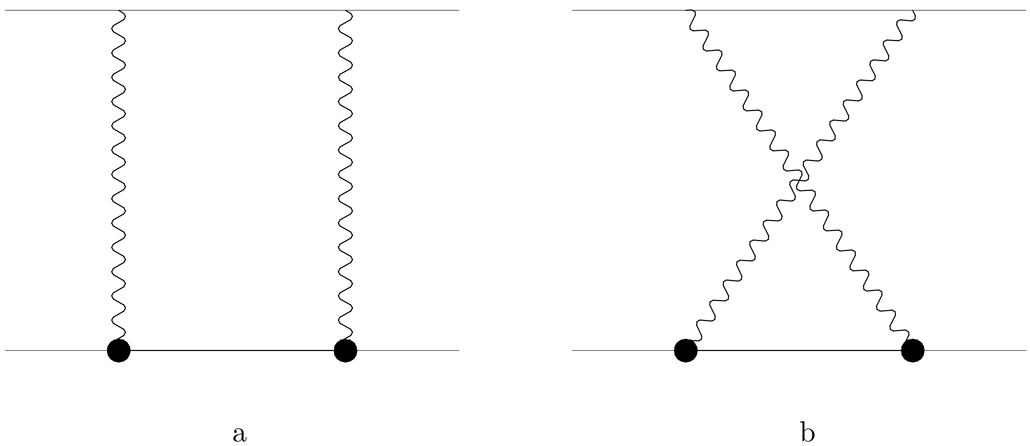


FIG. 2: Proton structure corrections of order $(Z\alpha)^5$. Bold circle in the diagram represents the proton vertex operator.

To construct the quasipotential corresponding to these diagrams we write the proton tensor:

$$M^{(p)}_{\mu\nu}=\bar u(q_2)\left[\gamma_\mu F_1+\frac{i}{2m_2}\sigma_{\mu\omega}k^\omega F_2\right]\frac{\hat p_2-\hat k+m_2}{(p_2-k)^2-m_2^2+i0}\left[\gamma_\nu F_1-\frac{i}{2m_2}\sigma_{\nu\lambda}k^\lambda F_2\right]u(p_2), \tag{34}$$

where p_2, q_2 are four momenta of the proton in initial and final states. The construction of the potential can be essentially simplified using the projection operators for the system muon-proton on the states with definite spin:

$$\hat\pi(^1S_0)=[u(p_2)\bar v(p_1)]_{S=0}=\frac{(1+\gamma^0)}{2\sqrt{2}}\gamma_5,\quad \hat\pi(^3S_1)=[u(p_2)\bar v(p_1)]_{S=1}=\frac{(1+\gamma^0)}{2\sqrt{2}}\hat\epsilon. \tag{35}$$

where ϵ^μ is the polarization vector of the state with the spin 1. Neglecting relative motion momenta of the particles in the initial and final states we obtain

$$\Delta E^{HFS}_{str}=E^F(2S)\frac{Z\alpha m_1m_2}{8\pi(1+\kappa)}\delta_{l0}\int\frac{id^4k}{\pi^2(k^2)^2}\left[\frac{16k^6k_0^2}{m_2^2}F_2^2+\frac{32k^8}{m_2^2}F_2^2-64k^2k_0^4F_2^2+\right. \tag{36}$$

$$\left.+16k^4k_0^2F_1^2+128k^4k_0^2F_1F_2+64k^4k_0^2F_2^2+32k^6F_1^2+64k^6F_1F_2\right]\frac{1}{(k^4-4m_1^2k_0^2)(k^4-4m_2^2k_0^2)}.$$

Transforming the integration in Eq.(36) to the Euclidean space

$$\int d^4k=4\pi\int k^3dk\int\sin^2\phi d\phi,\quad k_0=k\cos\phi, \tag{37}$$

we make analytical integration over the angle ϕ and present the correction (36) as the one-dimensional integral over the variable k:

$$\Delta E^{HFS}_{str}=-E^F(2S)\frac{Z\alpha}{8\pi(1+\kappa)}\delta_{l0}\int_0^\infty\frac{dk}{k}V(k), \tag{38}$$

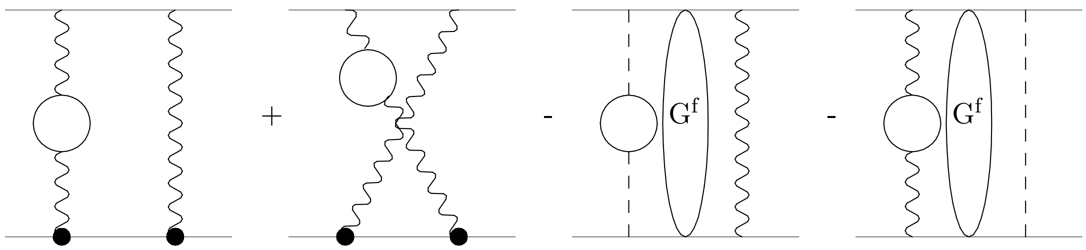


FIG. 3: Vacuum polarization and proton structure corrections of order $\alpha(Z\alpha)^5$. Dashed line in the diagram represents the Coulomb photon.

$$V(k)=\frac{2F_2^2k^2}{m_1m_2}+\frac{\mu}{(m_1-m_2)k(k+\sqrt{4m_1^2+k^2})}\left[-128F_1^2m_1^2-128F_1F_2m_1^2+16F_1^2k^2+\right.$$
$$\left.+64F_1F_2k^2+16F_2^2k^2+\frac{32F_2^2m_1^2k^2}{m_2^2}+\frac{4F_2^2k^4}{m_1^2}-\frac{4F_2^2k^4}{m_2^2}\right]+\frac{\mu}{(m_1-m_2)k(k+\sqrt{4m_2^2+k^2})}\times$$
$$\times\left[128F_1^2m_2^2+128F_1F_2m_2^2-16F_1^2k^2-64F_1F_2k^2-48F_2^2k^2\right].$$

To cancel infrared divergence in Eq.(38) it is necessary to add the contribution of the iteration term of the quasipotential (10) in the HFS of the μp:

$$\Delta E^{HFS}_{iter,str}=-<V_{1\gamma}\times G^f\times V_{1\gamma}>^{HFS}_{str}=-\frac{8}{3}\frac{\mu^4(Z\alpha)^5(1+\kappa)}{m_1m_2\pi}\int_0^\infty\frac{dk}{k^2},\tag{39}$$

where the angular brackets represent the averaging of the interaction operator over the Coulomb wave functions and the index HFS shows the hyperfine part in the iteration term of the quasipotential (8). The sum of the expressions (38) and (39) coincides with the result of Ref.[27]. The integration in Eqs.(38) and (39) was done by means of the parameterization of the proton electromagnetic form factors obtained from the analysis of elastic lepton-nucleon scattering [19]. Numerically the proton structure correction of order $(Z\alpha)^5$ is equal

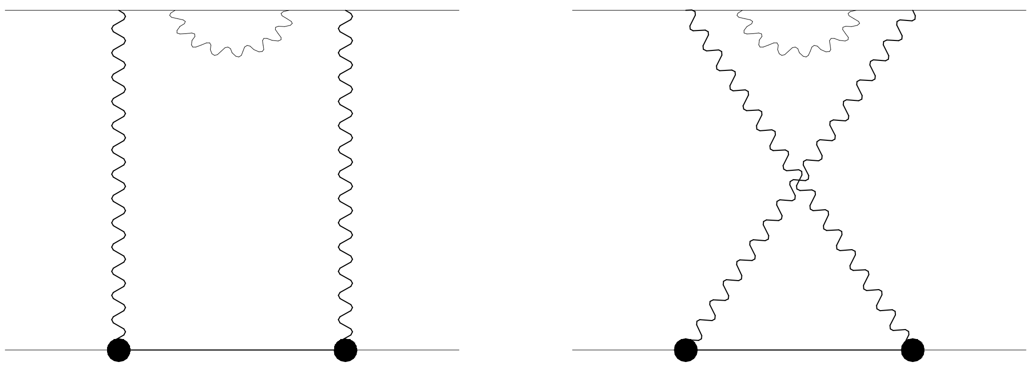
$$\Delta E^{HFS}_{str}+\Delta E^{HFS}_{iter,\ str}=-0.1518\ meV\tag{40}$$

Moreover, the effects of the proton structure must be taken into account carefully in the amplitudes of higher order over α shown in Fig.3.

The contributions of the diagrams (a) and (b) in Fig.3 to the potential can be found as for the amplitudes in Fig.2. taking into account the transformation of one exchange photon propagator as in Eq.(9). Corresponding correction to the HFS of the energy level is equal

$$\Delta E^{HFS}_{str,VP}=-E^F(2S)\frac{Z\alpha}{8\pi(1+\kappa)}2\frac{\alpha}{\pi}\int_0^1\frac{v^2\left(1-\frac{v^2}{3}\right)dv}{k^2(1-v^2)+4m_e^2}\int_0^\infty dkV_{VP}(k),\tag{41}$$

where the potential $V_{VP}(k)$ differs from $V(k)$ in the relation (38) only by the factor k^2. Despite of the finiteness of the integral (41) the amplitude terms of the quasipotential in Fig.3 (a), (b) must be completed by two iteration terms shown in Fig. 3 (c), (d). First addendum $<V^c\times G^f\times\Delta V^{HFS}_{VP}>$ of order $\alpha(Z\alpha)^4$ must be subtracted because the 2γ amplitudes (a)



a b

FIG. 4: Proton structure and muon self-energy effects of order $\alpha(Z\alpha)^5$.

and (b) in Fig.3 produce lower order contribution. Second term $< V^c_{VP}\times G^f\times V^{HFS}_{1\gamma} >$ which is also of order $\alpha(Z\alpha)^4$ has the structure similar to Eq.(29) of the second order of the perturbative series. The contributions of discussed iteration terms to the HFS of the μp coincide:

$$\Delta E^{HFS}_{iter,VP+str}=-2< V^c\times G^f\times \Delta V^{HFS}_{VP} >^{HFS}=-2< V^c_{VP}\times G^f\times \Delta V^{HFS}_{1\gamma} >^{HFS}= \quad (42)$$

$$=-E^F(2S)\frac{4(Z\alpha)\mu\alpha}{m_e\pi^2}\int_0^\infty dk\int_0^1\frac{v^2\left(1-\frac{v^2}{3}\right)dv}{k^2(1-v^2)+1}.$$

Numerical value of the proton structure and vacuum polarization effects of the 2γ amplitudes

$$\Delta E^{HFS}_{VP,str}+2\Delta E^{HFS}_{iter,VP+str}=-0.0026\ meV. \quad (43)$$

Hadronic vacuum polarization contribution to the HFS in the μp was studied in Ref.[41]. Here we present it in the different form using the expressions (38) and (41):

$$\Delta E^{HFS}_{HVP}=-E^F(2S)\frac{\alpha(Z\alpha)}{4\pi^2(1+\kappa)}\int_{4m_\pi^2}^\infty\frac{\rho(s)ds}{k^2+s}\int_0^\infty dkV_{VP}(k). \quad (44)$$

Dividing the integration range over s on the intervals where the cross section of the e^+e^- annihilation into hadrons ($\rho(s)=\sigma^h(e^+e^-\to hadrons)/3s\sigma_{\mu\mu}$) is known from the experiment [42] we can make numerical integration in Eq.(44). Numerical value coincides with the result obtained in Ref.[41]:

$$\Delta E^{HFS}_{HVP}=0.0005\ meV. \quad (45)$$

V. PROTON STRUCTURE EFFECTS, SELF ENERGY AND VERTEX CORRECTIONS OF ORDER $\alpha(Z\alpha)^5$

There exists real number of important contributions of order α^6 which are presented in Fig.4,5. Radiative corrections of these amplitudes including recoil effects were studied earlier both in the Lamb shift and HFS of the hydrogen-like systems [17, 43, 44]. Radiative photons were taken in the Fried-Yennie (FY) gauge [45, 46, 47] where the mass shell amplitudes don't

contain infrared divergences. Infrared finiteness of the Feynman diagrams in this gauge gives the possibility to make standard subtraction on the mass shell without introducing the photon mass. Let us consider radiative corrections which are determined by the self-energy insertions in the muon line. The renormalizable mass operator in the FY gauge is equal [17]:

$$\Sigma^R(p)=\frac{\alpha}{\pi}(\hat{p}-m)^2\int_0^1 dx\frac{-3\hat{p}x}{m_1^2x+(m_1^2-p^2)(1-x)}. \tag{46}$$

Making the insertion (46) in the lepton tensor of the two-photon exchange diagrams and using the projection operators (35) we can construct the hyperfine splitting part of the quasipotential for the amplitudes in Fig. 4. In this case as before the vertex of the proton-photon interaction is determined by electric and magnetic form factors because the typical loop momenta are of order the muon mass. The contraction of the lepton and proton tensors over the Lorentz indices and the Dirac γ matrix trace calculation were made in the system Form [48]. In the Euclidean space of the variable k we can present the correction to the HFS of the muonic hydrogen as follows:

$$\Delta E^{HFS}_{2\gamma,SE}=\frac{(Z\alpha)^5\mu^3}{8\pi^2}\delta_{l0}\frac{\alpha}{\pi}\int_0^1 xdx\int_0^\infty kdk\int_0^\pi \sin^2\phi d\phi V_{SE}(k,\phi,x), \tag{47}$$

$$V_{SE}(k,\phi,x)=\frac{1}{(k^2+4m_2^2\cos^2\phi)[(xm_1^2+\bar{x}k^2)^2+4m_1^2\bar{x}^2k^2\cos^2\phi]}\times \tag{48}$$

$$\times\Biggl\{-\frac{4m_1^2}{m_2^2}k^2F_2^2(x+6\bar{x})\cos^2\phi-\frac{8m_1^2}{m_2^2}k^2xF_2^2+16m_1^2F_2\cos^4\phi(4F_1\bar{x}-F_2x-2F_2\bar{x})+$$
$$+16m_1^2\cos^2\phi(F_1^2x+6F_1^2\bar{x}+4F_1F_2x+8F_1F_2\bar{x}+F_2^2x+2F_2^2\bar{x})+$$
$$+32m_1^2xF_1(F_1+F_2)-\frac{4k^4}{m_2^2}F_2^2\bar{x}\cos^2\phi-\frac{8k^4}{m_2^2}F_2^2\bar{x}-16k^2F_2^2\bar{x}\cos^4\phi+$$
$$+16k^2\bar{x}\cos^2\phi(F_1^2+4F_1F_2+F_2^2)+32k^2F_1\bar{x}(F_1+F_2)\Biggr\}.$$

After analytical integration over the angle ϕ we present the contribution (47) in the integral form which was used for numerical calculation:

$$\Delta E^{HFS}_{2\gamma,SE}=E^F(2S)\frac{m_1m_2\alpha(Z\alpha)}{\pi^2(1+\kappa)}\delta_{l0}\int_0^1 xdx\int_0^\infty dk\Biggl\{\left[-\frac{8F_2^2k^2}{m_2^2}+32F_1(F_1+F_2)\right]\frac{1}{h_1(k,x)}+ \tag{49}$$
$$+\left[-\frac{k^3F_2^2}{m_2^4}-\frac{6m_1^2k^3F_2^2\bar{x}}{m_2^4(xm_1^2+\bar{x}k^2)}+\frac{4k}{m_2^2}(F_1^2+4F_1F_2+F_2^2)\right]\left(\frac{1}{h_2(k,x)}-\frac{k}{h_1(k,x)}\right)+$$
$$\left[\frac{2km_1^2}{m_2^2}F_2(2F_1+F_2)\bar{x}-\frac{kF_2^2}{m_2^2}(xm_1^2+\bar{x}k^2)\right]\left[\frac{2}{h_2^2(k,x)}-\frac{k^2}{m_2^2(xm_1^2+\bar{x}k^2)}\left(\frac{1}{h_2(k,x)}-\frac{k}{h_1(k,x)}\right)\right],$$
$$h_1(k,x)=k\sqrt{4m_1^2\bar{x}^2k^2+(xm_1^2+\bar{x}k^2)^2}+(xm_1^2+\bar{x}k^2)\sqrt{4m_2^2+k^2},$$
$$h_2(k,x)=\sqrt{4m_1^2\bar{x}^2k^2+(xm_1^2+\bar{x}k^2)^2}+(xm_1^2+\bar{x}k^2).$$

Numerical value is equal

$$\Delta E^{HFS}_{2\gamma,SE}=0.0010\ meV. \tag{50}$$

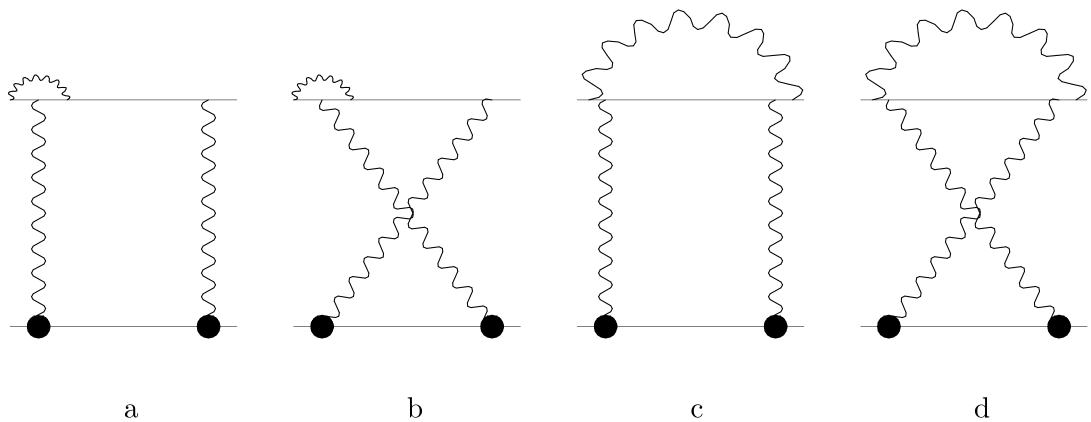


FIG. 5: Proton structure and muon vertex effects of order $\alpha(Z\alpha)^5$.

Let us consider calculation of the vertex corrections. The renormalizable expression of the one-particle vertex operator in the FY gauge was obtained in Ref. [49] ($p_1^2 = m_1^2$):

$$\Lambda_\mu^R(p, p-k) = \frac{\alpha}{4\pi}\int_0^1 dx \int_0^1 dz \left[\frac{F_\mu^{(1)}}{\Delta} + \frac{F_\mu^{(2)}}{\Delta^2}\right], \tag{51}$$

where $\Delta = m_1^2 x + 2pk(1-x)z - k^2 z(1-xz)$, the functions $F_\mu^{(1)}$, $F_\mu^{(2)}$ were determined in Ref.[49]. The lepton tensor can be divided into two parts:

$$M_{\mu\nu}^{(l)(1)} = \frac{\bar v(p_1)F_\nu^{(1)}(-\hat p_1 - \hat k + m_1)\gamma_\mu v(q_1)(k^2 - 2k^0 m_1)[m_1^2 x - k^2 z(1-xz) + 2m_1 k^2 \bar x^2]}{(k^4 - 4k_0^2 m_1^2)[(m_1^2 x - k^2 z(1-xz))^2 - 4m_1^2 k_0^2 \bar x^2 z^2]}, \tag{52}$$

$$M_{\mu\nu}^{(l)(2)} = \frac{\bar v(p_1)F_\nu^{(2)}(-\hat p_1 - \hat k + m_1)\gamma_\mu v(q_1)(k^2 - 2k^0 m_1)[m_1^2 x - k^2 z(1-xz) + 2m_1 k^2 \bar x^2]^2}{(k^4 - 4k_0^2 m_1^2)[(m_1^2 x - k^2 z(1-xz))^2 - 4m_1^2 k_0^2 \bar x^2 z^2]^2}. \tag{53}$$

Keeping for the simplicity the main contribution over m_1/m_2 we write this type vertex corrections as follows:

$$\Delta E^{HFS}_{2\gamma, vert\ 1} = -E^F(2S)\left(\frac{\alpha}{\pi}\right)^2 \frac{8m_1m_2}{(1+\kappa)\pi}\int_0^1 dx \int_0^1 dz \int_0 \pi \sin^2\phi d\phi \int_0^\infty k dk \times \tag{54}$$

$$\times \frac{V_1(x,k,\phi)[F_1(F_1+F_2) - (1+\kappa)]}{(k^2 + 4m_1^2\cos^2\phi)(k^2 + 4m_2^2\cos^2\phi)\,[[m_1^2 x + k^2 z(1-zx)]^2 + 4m_1^2 k^2 \cos^2\phi \bar x^2 z^2]},$$

$$V_1(x,k,\phi) = -2m_1^4 x^2(1-x) + k^2 m_1^2(6x^3z^2 - 8x^2z^2 - 3x^2z + 8xz - 3x) + \tag{55}$$

$$+k^4(4x^3z^4 - 6x^2z^4 - 5x^2z^3 + 12xz^3 - 2xz^2 - 6z^2 + 3z),$$

$$\Delta E^{HFS}_{2\gamma, vert\ 2} = -E^F(2S)\left(\frac{\alpha}{\pi}\right)^2 \frac{32m_1^3m_2}{(1+\kappa)\pi}\int_0^1 x(1-x)dx \int_0^1 dz \int_0 \pi \sin^2\phi d\phi \int_0^\infty k^3 dk \times \tag{56}$$

$$\times \frac{V_2(x,k,\phi)F_1(F_1+F_2)}{(k^2 + 4m_1^2\cos^2\phi)(k^2 + 4m_2^2\cos^2\phi)\,[[m_1^2 x + k^2 z(1-zx)]^2 + 4m_1^2 k^2 \cos^2\phi \bar x^2 z^2]^2},$$

$$V_2(x,k,\phi)=m_1^4x^2z(2z-1)-k^2m_1^2xz^2(4xz^2-2xz-4z+2)+ \\ +k^4z^3(2x^2z^3-x^2z^2-4xz^2+2xz+2z-1). \quad (57)$$

The iteration contribution is equal

$$\Delta E^{HFS}_{iter,\ 2\gamma\ vert}=<V_{1\gamma}\times G^f\times V_{1\gamma}>^{HFS}_{vert}=F^F\left(\frac{\alpha}{\pi}\right)^2\int_0^1 dz\int_0^1 dx\int_0^\infty dk\frac{4\mu}{k^2}, \quad (58)$$

After analytical integration in Eqs.(54) and (56) over the angle ϕ and the subtraction (58) (one photon is the Coulomb-like and the other one contains the hyperfine part of the potential with the value of magnetic form factor at zero point) we have the expressions of the diagrams (a) and (b) in Fig. 5:

$$\Delta E^{HFS}_{2\gamma,\ vert}=-E^F(2S)\left(\frac{\alpha}{\pi}\right)^2\int_0^1 dx\int_0^1 dz\int_0^\infty dk\Biggl\{\frac{F_1(F_1+F_2)}{8k(1+\kappa)m_1^3m_2\bar x^2z^2}\Bigl[-2m_1^4x^2\bar x+k^2m_1^2x\times \quad (59)$$

$$\times(6x^2z^2-8xz^2-3xz+8z-3)+k^4z(4x^3z^3-6x^2z^3-5x^2z^2+12xz^2-2xz-6z-3)\Bigr]\times$$

$$\left[-\frac{\sqrt{1+b^2}}{b(a^2-b^2)(b^2-c^2)}+\frac{\sqrt{1+a^2}}{a(a^2-b^2)(a^2-c^2)}+\frac{\sqrt{1+c^2}}{c(b^2-c^2)(a^2-c^2)}\right]+\frac{F_1(F_1+F_2)x}{2(1+\kappa)m_1^3m_2k\bar x^3z^4}\times$$

$$\Bigl[m_1^4x^2z(2z-1)-2k^2m_1^2xz^2(2xz^2-xz-2z+1)+k^4z^3(2x^2z^3-x^2z^2-4xz^2+2xz+2z-1)\Bigr]\times$$

$$\times\Biggl[-\frac{\sqrt{1+b^2}}{b(a^2-b^2)(b^2-c^2)^2}+\frac{\sqrt{1+a^2}}{a(a^2-b^2)(a^2-c^2)}+\frac{1}{2c\sqrt{1+c^2}(b^2-c^2)(a^2-c^2)}-$$

$$-\frac{\sqrt{1+c^2}}{2c^3(b^2-c^2)(a^2-c^2)}+\frac{\sqrt{1+c^2}}{c(b^2-c^2)(a^2-c^2)}+\frac{\sqrt{1+c^2}}{c(b^2-c^2)(a^2-c^2)^2}\Biggr]+\frac{4\mu}{k^2}\Biggr\},$$

$$a^2=\frac{k^2}{4m_1^2},\quad b^2=\frac{k^2}{4m_2^2},\quad c^2=\frac{[m_1^2x+k^2z(1-xz)]^2}{4m_1^2k^2\bar x^2z^2}. \quad (60)$$

Numerical value of the vertex correction (59) is equal

$$\Delta E^{HFS}_{2\gamma,\ vert}=-0.0018\ meV \quad (61)$$

Next vertex type diagram with one rounded photon and two exchanged photons is the diagram of the "jellyfish" type. Its contribution to the energy spectrum is of order $\alpha(Z\alpha)^5$. At small loop momenta this diagram gives the finite answer in the FY gauge. The lepton tensor relating to the diagrams (c) and (d) in Fig.5 was obtained in Ref.[44]:

$$L^{(\mu)}_{\mu\nu}=\frac{\alpha}{4\pi}\int_0^1 xdx\int_0^1(1-z)dz\sum_{n=1}^3\frac{M^{(n)}_{\mu\nu}}{\Delta^n}, \quad (62)$$

where Δ has the form as in Eq.(51). The tensor functions $M^{(n)}_{\mu\nu}$ are written explicitly in Ref.[44]. The character of further transformations of the amplitudes (c), (d) in Fig.5 to construct the HFS part of the potential is the same as for other amplitudes shown in Figs.4,5. Omitting the details of such transformations which were carried out by means of

analytical system Form [48] we write here three contributions to the HFS corresponding to the functions $M^{(n)}_{\mu\nu}$ in the leading order over the ratio m_1/m_2:

$$\Delta E^{HFS}_{1,\ jellyfish}=-\frac{8\alpha(Z\alpha)^5\mu^3\delta_{l0}}{\pi^3}\int_0^1 xdx\int_0^1(1-z)(1-3xz)\int_0^\infty kdkF_1(F_1+F_2)\times \tag{63}$$

$$\times\int_0^\pi\frac{\sin^2\phi d\phi}{(k^2+4m_2^2\cos^2\phi)}\frac{[m_1^2x+k^2z(1-xz)]}{[m_1^2x+k^2z(1-xz)]^2+4m_1^2k^2\cos^2\phi\bar{x}^2z^2},$$

$$\Delta E^{HFS}_{2,\ jellyfish}=-\frac{16\alpha(Z\alpha)^5\mu^3\delta_{l0}}{3\pi^3}\int_0^1 xdx\int_0^1(1-z)dz\int_0^\infty kdkF_1(F_1+F_2)\times \tag{64}$$

$$\times\int_0^\pi\frac{\sin^2\phi d\phi}{(k^2+4m_2^2\cos^2\phi)}\frac{[m_1^2x+k^2z(1-xz)]^2[k^2xz^2(1-xz)+m_1^2(x^2z+2xz-x-3z)]}{\{[m_1^2x+k^2z(1-xz)]^2+4m_1^2k^2\cos^2\phi\bar{x}^2z^2\}^2},$$

$$\Delta E^{HFS}_{3,\ jellyfish}=\frac{64\alpha(Z\alpha)^5\mu^3\delta_{l0}}{3\pi^3}\int_0^1 xdx\int_0^1(1-z)z^2dz\int_0^\infty k^3dkm_1^2F_1(F_1+F_2)\times \tag{65}$$

$$\times(x+xz-x^2z-1)\int_0^\pi\frac{\sin^2\phi d\phi}{(k^2+4m_2^2\cos^2\phi)}\frac{[m_1^2x+k^2z(1-xz)]^3}{\{[m_1^2x+k^2z(1-xz)]^2+4m_1^2k^2\cos^2\phi\bar{x}^2z^2\}^3}.$$

The integration over the angle ϕ can be done in Eqs.(63)-(65) analytically. Omitting intermediate expressions we can write final numerical result to the HFS of the μp:

$$\Delta E^{HFS}_{jellyfish}=\sum_{n=1}^3\Delta E^{HFS}_{n,\ jellyfish}=0.0005\ meV. \tag{66}$$

In the point-like proton approximation when the nucleus form factors entering the Feynman amplitudes in Fig. 4,5 are changed on their values at $k^2=0$ ($F_1(0)=1$, $F_2(0)=\kappa$) the contributions (63)-(65) will increase twofold.

VI. CONCLUSION

The calculation of different quantum electrodynamical effects, effects of the proton structure and polarizability, the hadron vacuum polarization to the HFS of muonic hydrogen is performed in this work. The corrections of orders α^5 and α^6 are considered. Working with the vacuum polarization diagrams we take into account that the ratio $\mu\alpha/m_e$ is very close to 1 and don't increase the order of corresponding contributions. Obtained numerical results are presented in the Table 1. We include here also QED corrections to the Fermi energy which are determined by the muon anomalous magnetic moment $a_\mu E^F(2S)$ [17] (experimental value of the muon anomalous magnetic moment $a_\mu^{exp}=11659203(8)\times 10^{-10}$ [50] is used), the Breit relativistic correction of order $(Z\alpha)^6$ [51], the proton structure corrections of order $(Z\alpha)^6\ln(Z\alpha)^2$ [7], the hadron vacuum polarization contribution [41] and the proton polarizability correction [16], the weak interaction contribution due to the Z boson exchange [52].

Let us point out some peculiarities of this investigation.

1.The effects of the vacuum polarization play very important role in the case of the muonic hydrogen. They lead to essential modification of the spin-dependent part of the quasipotential of the one-photon interaction.

TABLE II: Corrections of orders α^5, α^6 to the $2S$ state HFS in the muonic hydrogen.

Contribution to HFS of μp	Numerical value in meV	Reference
The Fermi energy $E^F(2S)$	22.8054	[17], (12)
Muon AMM correction $a_\mu E^F(2S)$ of order α^5, α^6	0.0266	[17]
Relativistic correction $\frac{17}{8}(Z\alpha)^2 E^F(2S)$ of order α^6	0.0026	[51]
Relativistic and radiative recoil corrections with the account proton AMM of order α^6	0.0018	[38]
One-loop electron vacuum polarization contribution of 1γ interaction of orders α^5, α^6	0.0482	(18)
One-loop muon vacuum polarization contribution of 1γ interaction of order α^6	0.0004	(19)
Vacuum polarization corrections of orders α^5, α^6 in the second order of perturbative series	0.0746	(30)+(33)
Proton structure corrections of order α^5	-0.1518	[27], (40)
Proton structure corrections of order α^6	-0.0017	[7]
Electron vacuum polarization contribution+ proton structure corrections of order α^6	-0.0026	(43)
Two-loop electron vacuum polarization contribution of 1γ interaction of order α^6	0.0003	(21)+(24)
Muon self energy + proton structure correction of order α^6	0.0010	(50)
Vertex corrections + proton structure corrections of order α^6	-0.0018	(61)
"Jellyfish" diagram correction + proton structure corrections of order α^6	0.0005	(66)
HVP contribution of order α^6	0.0005	(45)
Proton polarizability contribution of order α^5	0.0105	[16]
Weak interaction contribution	0.0003	[52]
Summary contribution	22.8148 ± 0.0078	

2. The proton structure effects are taken into account consistently in the loop amplitudes by means of electromagnetic form factors. The point-like proton approximation gives essentially increased results (approximately twofold).

3. The calculation of the muon self-energy and vertex corrections of order $\alpha(Z\alpha)^5$ is done on the basis of the expressions for the lepton factors in the amplitude terms of the quasipotential obtained by Eides, Grotch and Shelyuto. We supplemented these relations by the subtraction of the iteration terms of the potential.

Total value of the $2S$ state HFS in the muonic hydrogen shown in the Table 1 can be considered as definite guide for the future experiment which is prepared [20]. Numerical values of the corrections were obtained with the accuracy 0.0001 meV. Theoretical error connected with the uncertainties of fundamental physical constants (fine structure constant,

the proton magnetic moment etc.) entering the Fermi energy compose the value near 10^{-6} meV. Other source of theoretical uncertainty is connected with the corrections of higher order. Its estimation can be found from the leading correction of the next order on α and m_1/m_2 in the form: $\alpha(Z\alpha)^2\ln(Z\alpha)^2 E^F(2S)/\pi \approx 0.00003$ meV (the value of fine structure constant is $\alpha^{-1} = 137.03599976(50)$ [1]). Further improvement of theoretical result presented in the Table 1 is connected first of all with the corrections on the proton structure and polarizability which give the theoretical error near 34×10^{-5} ppm. The most part of this error is determined by the proton structure corrections of order $(Z\alpha)^5$ (the Zemach correction). So, the measurement of the hyperfine splitting of the levels $1S$ and $2S$ in the muonic hydrogen with the accuracy 30 ppm will lead to more accurate value (with relative error 10^{-3}) for the Zemach radius which than can be used for the improvement of theoretical result for the ground state hydrogen hyperfine structure and more reliable estimation of the proton polarizability effect.

Another important quantity regarding to the hyperfine structure of the muonic hydrogen is the Sternheim's hyperfine splitting interval [30]. It doesn't contain the proton structure and polarizability effects of the leading order and provides additional test of quantum electrodynamics for the hydrogen atom. Accounting numerical results of this work and Ref.[29] we can write numerical value for this interval in the form:

$$[8\Delta E^{HFS}(2S) - \Delta E^{HFS}(1S)] = -0.120\ meV. \tag{67}$$

This result is valid with a precision 10^{-6} due to the cancellation of the proton structure and polarizability corrections. The increase the number of the tasks due to excited states of simple atoms [53] and the inclusion new simple atoms where the hyperfine structure of the energy spectrum is studied will decrease the uncertainties in the determination of physical fundamental parameters and increase the accuracy for the check of the Standard Model in low energy physics.

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[1] P.J.Mohr, B.N.Taylor, *Rev. Mod. Phys.* **72**, 351 (2000).
[2] A.C.Zemach, *Phys. Rev.* **104**, 1771 (1956).
[3] H. Hellwig, R.F.C.Vessot, M.W.Levine, et al, *IEEE Trans.* **IM-19**, 200 (1970).
[4] F.Guerin, *Nuovo Cimento* **A50**, 1 (1967).
[5] G.M.Zinoviev, B.V.Struminskii, R.N.Faustov et al., *Yad. Fiz.* **11**, 1284 (1970).
[6] A.I.Milshtein, S.S.Petrosyan, I.B.Khriplovich, *JETP*, **109** 1146 (1996).
[7] S.G.Karshenboim, *Phys. Lett.* **A225**, 97 (1997).
[8] J.L.Friar, *Can. J. Phys.* **80**, 1337 (2002).
[9] A.Pineda, *Phys. Rev.* **C67**, 025201 (2003).
[10] R.N.Faustov, A.P.Martynenko, *Phys. Rev.* **A67**, 052506 (2003).
[11] S.D.Drell, J.D.Sullivan, *Phys. Rev.* **154**, 1477 (1967).

[12] J.Bernabeu, T.E.O.Ericson, *Z. Phys.* **A309**, 213 (1983).
[13] G.T.Bodwin, D.R.Yennie, *Phys. Rev.* **D37**, 498 (1988).
[14] A.P.Martynenko, R.N.Faustov, *Yad. Fiz.*, **63** 915 (2000).
[15] K.Pachucki, Phys. Rev. **A60**, 3593 (1999).
[16] E.V.Cherednikova, R.N.Faustov, A.P.Martynenko, *Nucl. Phys.* **A703**, 365 (2002).
[17] M.I.Eides, H.Grotch, V.A.Shelyuto, *Phys. Rep.* **342**, 62 (2001).
[18] D.D.Bakalov, E.Milotti, C.Rizzo et al., in: *Proc. 3rd Int. Symp. on Weak and Electromagnetic Int. in Nuclei* (WEIN-92), 656 (1992)
[19] G.G.Simon, Ch.Schmitt, F.Borkowski et al., *Nucl. Phys.* **A333**, 381 (1980).
[20] A.Adamczak, D.Bakalov, K.Bakalova et al., *Hyperfine Interactions* **136**, 1 (2001).
[21] A.Dupays, A.Beswick, B.Lepetit et al., *Phys. Rev.* **A68**, 052503 (2003).
[22] A.V.Volotka, V.M.Shabaev, G.Plunien, G.Soff, E-preprint hep-ph/0405118.
[23] J.L.Friar, I.Sick, *Phys. Let.* **B579** (2004) 285.
[24] S.J.Brodsky, C.E.Carlson, J.R.Hiller, D.S.Hwang, Preprint SLAC-PUB-10565; hep-ph/0408131.
[25] M.M.Sternheim, *Phys. Rev.* **138**, B430 (1965).
[26] E.Borie, G.A.Rinker, *Rev. Mod. Phys.* **54**, 67 (1982).
[27] K.Pachucki, *Phys. Rev.* **A53**, 2092 (1996).
[28] A.Veitia, K.Pachucki, *Phys. Rev.* **A69**, 042501, (2004).
[29] R.N.Faustov, A.P.Martynenko, *JETP* **98**, 39 (2004).
[30] M.M.Sternheim, *Phys. Rev.* **130**, 211 (1963).
[31] F.Kottmann, F.Biraben, C.A.N. Conde et al., in: G.Cantatore (Ed.), *Quantum Electrodynamics and Physics of the Vacuum*, QED 2000 Second Workshop Proc. New York, AIP Conf. Proc., **564**, 13 (2001);
R.Pohl, F.Biraben, C.A.N.Conde et al., *Hyperfine Interactions* **127**, 161 (2000).
[32] A.P.Martynenko, R.N.Faustov, *Teor. Mat. Fiz.* **64**, 179 (1985).
[33] V.B.Berestetskii, E.M.Lifshits, L.P.Pitaevskii, *Quantum Electrodynamics*, M., Nauka (1980).
[34] R.N.Faustov, *Fiz. Elem. Chast. i Atom. Yadra* **3**, 238 (1972).
[35] M.I.Eides, H.Grotch, V.A.Shelyuto, *Phys. Rev.* **D65**, 013003 (2001).
[36] S.A.Zapryagaev, N.L.Manakov, V.G.Palchikov, *Theory of multicharged ions with one and two electrons*, M., Energoatomizdat, 1985.
[37] V.G.Ivanov, S.G.Karshenboim, *JETP* **109**, 1219 (1996).
[38] G.T.Bodwin, D.R.Yennie, M.A.Gregorio, *Rev. Mod. Phys.* **57**, 723 (1985).
[39] T.Kinoshita, M.Nio, *Phys. Rev.* **D53**, 4909 (1996).
[40] A.P.Martynenko, R.N.Faustov, *JETP* **115**, 1221 (1999).
[41] A.P.Martynenko, R.N.Faustov, *Yad. Fiz* **61**, 471 (1998).
[42] R.R.Akhmetshin et al., *Phys. Lett.* **B476**, 33 (2000);
R.R.Akhmetshin et al., *Phys. Lett.* **B509**, 217 (2001).
[43] M.I.Eides, S.G.Karshenboim, V.A.Shelyuto, *Ann. Phys.* **205**, 231 (1991).
[44] M.I.Eides, H.Grotch, V.A.Shelyuto, *Phys. Rev.* **A63**, 052509 (2001).
[45] A.A.Abrikosov, *JETP* **30**, 96 (1956).
[46] L.P.Gorkov, *JETP* **30**, 790 (1956).
[47] H.M.Fried, D.R.Yennie, *Phys. Rev.* **112**, 1391 (1958).
[48] J.A.M.Vermaseren, *FORM*, *E-preprint* math-ph/0010025, 2000; http://www.nikhef.nl/∼form.
[49] M.I.Eides, V.A.Shelyuto, *Eur. Phys. Jour.* **C21**, 489 (2001).
[50] G.W.Bennett et al., *Phys. Rev. Lett.* **89**, 101804 (2002)

[51] G.Breit, *Phys. Rev.* **35**, 1447 (1930).
[52] M.I.Eides, *Phys. Rev.* **A53**, 2953 (1996).
[53] S.G.Karshenboim, V.G.Ivanov, *Eur. Phys. Jour.* **D19**, 13 (2002).